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The two-dimensional harmonic oscillator interacting with a wedge

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Abstract. The exact propagator of our dynamic system is presented and confirmed by expanding it in terms of the energy eigenfunctions and eigenvalues, which agree with those obtained from the corresponding Schrödinger equation. For the case of a rational wedge the propagator can be expressed as a sum over 'classical paths', but with the modified Van-Vleck formula. We also evaluate the density matrix and the partition function.

1. Introduction

The time-dependent propagator is the quantity $K(r'', t; r') = \langle r''|\exp(-\mathrm{i}tH/\hbar)|r'\rangle$ and is defined as the sum over history (path integral) by Feynman in his classic paper [1]. However, it is known in closed form in only a few cases [2-4]. Recently, Schulman [5] obtained the exact propagator for a particle subject to an infinite half-plane barrier. Wiegel and van Andel [6] extended Schulman's result by including a harmonic potential. Crandell [7] and Cécile Dewitt-Morette *et al* [8] evaluate the exact propagator for a particle interacting with a rational wedge. We have extended their results to a two-dimensional harmonic oscillator [9]. In this paper we generalize our results further.

In section 2, we solve the Schrödinger equation for a two-dimensional harmonic oscillator interacting with a wedge. In section 3 the proposed exact propagator is confirmed by expanding it in terms of the energy eigenfunctions and eigenvalues, which agree with the results in section 2. In section 4 we express the propagator as a sum over 'classical paths', but with the modified Van Vleck formula [10]. Finally, we evaluate the density matrix and the partition function in statistical mechanics.

2. Energy eigenfunctions and eigenvalues

For our dynamical system we assume that (a) the wedge is along the z axis, (b) the external angle of the wedge is $a\pi$ $(0 < a \leqslant 2)$ and (c) a harmonic potential is centred at the origin (see figure 1). The Schrödinger equation in polar coordinates is of the form

$$-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)\psi(r,\varphi)+\frac{\mu\omega^2}{2}r^2\psi(r,\varphi)=E\psi(r,\varphi) \qquad (1)$$

where μ is the mass and ω is angular frequency of the harmonic oscillator. The wavefunction must satisfy the following boundary conditions:

$$\psi(r,0)=\psi(r,a\pi)=0. \qquad (2)$$

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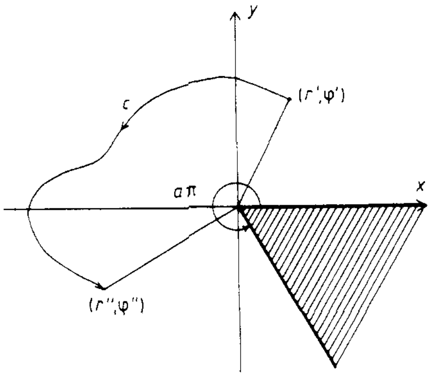

Figure 1. The edge of the wedge is the z axis. The harmonic force is centred at the origin of the (r, φ) plane. The path C represents one of the paths from the initial position (r', φ') to the final position (r'', φ'') during the time interval t.

We assume that the wavefunction has the form [11] $(\lambda = \mu\omega/\hbar)$

$$\psi(r, \varphi) = r^{l/a} \exp(-\lambda r^2/2) \sin(l\varphi/a) F(r) \qquad l = 1, 2, 3, \ldots \tag{3}$$

and the angular part $\sin(l\varphi/a)$ is introduced here in order to satisfy (2). Substituting (3) into (1), we have

$$\frac{d^2F(r)}{dr^2} + \left(\frac{2l/a+1}{r} - 2\lambda r\right)\frac{dF(r)}{dr} - [2\lambda(l/a+1) - k^2]F(r) = 0 \tag{4}$$

with $k^2 = 2mE/\hbar^2$. Now by using the variable $\xi = \lambda r^2$, (4) is transformed into the Kummer equation

$$\xi\frac{d^2F(\xi)}{d\xi^2} + [(l/a+1) - \xi]\frac{dF(\xi)}{d\xi} - \left(\tfrac{1}{2}(l/a+1) - \frac{k^2}{4\lambda}\right)F(\xi) = 0. \tag{5}$$

In order to obtain the solution regular at $r = 0$ (or $\xi = 0$), we find the degenerate hypergeometric function

$$F(\lambda r^2) = {}_1F_1(b, l/a+1; \lambda r^2) \tag{6}$$

and

$$b = \tfrac{1}{2}(l/a+1) - k^2/4\lambda. \tag{7}$$

For large r, F (λr^2) would diverge as $\exp(\lambda r^2)$, thus preventing normalization of the wavefunction. The wavefunction can be normalized only by choosing

$$b = -n_r \qquad n_r = 0, 1, 2, \ldots \tag{8}$$

which give the energy eigenvalues as

$$E_{n_r,l} = \hbar\omega(l/a + 1 + 2n_r). \tag{9}$$

With the help of (3) and (6), we obtain the energy eigenfunction as

$$\psi_{n_r,l}(r, \varphi) = C_{n_r,l}\, r^{l/a} \exp(-\lambda^2/2) \sin(l\varphi/a)\, {}_1F_1(-n_r, l/a+1; \lambda r^2) \tag{10}$$

with

$$C_{n_r,l} = \left((a\pi/2)\int_0^\infty r^{2l/a+1} \exp(-\lambda r^2)\, {}_1F_1^2(-n_r, l/a+1; \lambda r^2)\right)^{-1/2} \tag{11}$$

being the normalization constant. It is easy to see that from (9) there exist degenerate states when $l/a+2n_r$ is a positive integer. In other words, for a rational wedge, $a=n/m$ (n and m are positive integers), the energy spectrum (9) becomes

$$E_{n_r,l}=\hbar\omega(lm/n+1+2n_r)$$

which give the degenerate states if $2n_r=p-l(m/n)$ where p is an integer and is greater than 1. So far as we know, the integral involved in (11) has not been found in closed form. However, we will derive it in section 3. For later comparison, we evaluate the following simpler cases:

$$C_{0,1}=2[\lambda^{1/a+1}/a\pi\Gamma(1/a+1)]^{1/2} \qquad C_{0,2}=2[\lambda^{2/a+1}/a\pi\Gamma(2/a+1)]^{1/2}$$
$$C_{1,1}=2[\lambda^{1/a+1}/a\pi\Gamma(1/a+1)]^{1/2}(1/a+1) \tag{12}$$

where $\Gamma(\ )$ is the gamma function.

3. Exact time-dependent propagator

We propose that the exact time-dependent propagator is of the form

$$\begin{aligned} K_a(r'',\varphi'',t;r',\varphi') &= \left(\frac{\lambda}{2a\pi\mathrm{i}\sin(\omega t)}\right)\exp[\mathrm{i}\lambda(r'^2+r''^2)\cot(\omega t)/2] \\ &\times\sum_{l=-\infty}^{\infty}\{\exp[-\mathrm{i}l(\varphi''-\varphi')/a]-\exp[-\mathrm{i}l(\varphi''+\varphi')/a]\} \\ &\times I_{|l/a|}\left(\frac{\lambda r'r''}{\mathrm{i}\sin(\omega t)}\right). \end{aligned} \tag{13}$$

For the rational wedge, the above propagator has been demonstrated [9] by using the method of image and by computing the path integrals on a n-sheeted Riemann surface [8]. We will see that (13) is valid for the general case too. In order to do so, we rewrite (13) in the following form:

$$\begin{aligned} K_a(r'',\varphi'',t;r',\varphi') &= \frac{4\lambda}{a\pi}\exp\left(-\frac{\lambda}{2}(r'^2+r''^2)\right)\sum_{l=1}^{\infty}[\sin(l\varphi''/a)\sin(l\varphi'/a)]\exp(-\mathrm{i}\omega t) \\ &\times\left[[1-\exp(-2\mathrm{i}\omega t)]^{-1}\exp\left(-\lambda(r'^2+r''^2)\frac{\exp(-2\mathrm{i}\omega t)}{1-\exp(-2\mathrm{i}\omega t)}\right)\right. \\ &\times\left. I_{l/a}\left(2\lambda r'r''\frac{\exp(-\mathrm{i}\omega t)}{1-\exp(-2\mathrm{i}\omega t)}\right)\right] \end{aligned} \tag{14}$$

since

$$2\mathrm{i}\sin(\omega t)=[1-\exp(-2\mathrm{i}\omega t)]\exp(\mathrm{i}\omega t)$$

and

$$2\cos(\omega t)=[1+\exp(-2\mathrm{i}\omega t)]\exp(\mathrm{i}\omega t).$$

Using the identities [12]

$$(1-u)^{-1}\exp\left(-(c+d)\frac{u}{1-u}\right)I_\alpha\left(\frac{2(cdu)^{1/2}}{1-u}\right)$$
$$=(cdu)^{\alpha/2}\sum_{q=0}^{\infty}\frac{q!}{\Gamma(l/a+1+q)}L_q^\alpha(c)L_q^\alpha(d)u^q \qquad |u|<1 \tag{15}$$

$$L_q^\alpha(v)=\binom{q+\alpha}{q}{}_1F_1(-q,\alpha+1;v) \tag{16}$$

with $u=\exp(-2\mathrm{i}\omega t)$, $c=\lambda r'^2$, $d=\lambda r''^2$, $\alpha=l/a$ and $q=n_r$, we finally have

$$K_a(r'',\varphi'',t;r',\varphi')$$
$$=\frac{4\lambda}{a\pi}\exp\left(-\frac{\lambda}{2}(r'^2+r''^2)\right)\sum_{l=1}^{\infty}\sum_{n_r=0}^{\infty}\binom{n_r+l/a}{n_r}^2\frac{n_r!}{\Gamma(l/a+1+n_r)}$$
$$\times(\lambda r'^2)^{l/2a}(\lambda r''^2)^{l/2a}\sin(l\varphi''/a)\sin(l\varphi'/a)$$
$$\times{}_1F_1(-n_r,l/a+1;\lambda r'^2){}_1F_1(-n_r,l/a+1;\lambda r''^2)$$
$$\times\exp[-\mathrm{i}(l/a+1+2n_r)\omega t] \tag{17}$$

which gives the correct eigenfunctions and eigenvalues (see section 2) and the normalization constant (see (12) for comparison)

$$C_{n_r,l}=2\left(\frac{\lambda^{l/a+1}n_r!}{a\pi\Gamma(l/a+1+n_r)}\right)^{1/2}\binom{n_r+l/a}{n_r}.$$

As a by-product, we obtain the integral in (11) as

$$\int_0^\infty r^{2l/a+1}\,\mathrm{e}^{-\lambda r^2}{}_1F_1^2(-n_r,l/a+1;\lambda r^2)\,\mathrm{d}r=\binom{n_r+l/a}{n_r}^{-2}\left(\frac{a\pi\Gamma(l/a+1+n_r)}{2\lambda^{l/a+1}n_r!}\right). \tag{18}$$

We can see that the propagator (13) also satisfies the Dirichlet boundary condition by noting that one can obtain the second term in the braces from the first term by the transformation $\varphi'\to 2a\pi-\varphi'$, due to the wedge. Therefore, (13) is the exact propagator for a two-dimensional harmonic oscillator interacting with a wedge.

4. Sum over classical paths

Now we are going to discuss the case when the propagator can be expressed as the sum over classical paths. In order to investigate this we study the rational wedge case. Using the identity (see the appendix of [8] for deviations)

$$\tfrac{1}{2}J_0(\xi)+\sum_{l=1}^{\infty}\cos(lm\varphi/n)I_{lm/n}(\xi)$$
$$=\frac{1}{2m}\sum_{k=1}^{m}\Bigg\{\exp[-\mathrm{i}\xi\cos(\varphi+2\pi kn/m)]$$
$$\times\Bigg[1-\mathrm{i}\sum_{l=1}^{n-1}\cos\left(\frac{n-l}{n}(\varphi+2\pi kn/m)\right)\int_0^\xi[\mathrm{i}^{l/n}J_{-l/n}(u)$$
$$-(-\mathrm{i})^{l/n}J_{l/n}(u)]\exp[\mathrm{i}u\cos(\varphi+2\pi kn/m)]\,\mathrm{d}u\Bigg]\Bigg\}. \tag{19}$$

we obtain after lengthy but straightforward calculations the result

$$K_{n/m}(r'', \varphi'', t; r', \varphi')$$
$$= \left(\frac{\lambda}{2\pi \mathrm{i} \sin(\omega t)}\right) \frac{1}{n} \sum_{k=1}^{m} \left[\exp[\mathrm{i} S^{(n)}_{(m,k)}(\varphi'' - \varphi')/\hbar] \right.$$
$$\times \left(1 + \sum_{l=1}^{n-1} D^{(m,n)}_{l,k}(\varphi'' - \varphi')\right) - \exp[\mathrm{i} S^{(n)}_{(m,k)}(\varphi'' + \varphi')/\hbar]$$
$$\left. \times \left(1 + \sum_{l=1}^{n-1} D^{(m,n)}_{l,k}(\varphi' + \varphi'')\right)\right] \tag{20}$$

where the classical action is

$$[S^{(n)}_{(m,k)}(\varphi'' \mp \varphi')]$$
$$= \left(\frac{\lambda \hbar}{2 \sin(\omega t)}\right) [(r^2 + r''^2) \cos(\omega t)$$
$$- 2r'r'' \cos(\varphi'' \mp \varphi' + 2\pi k n/m)] \tag{21}$$

and the diffractive amplitude is ($z = \lambda r' r''/\sin(\omega t)$)

$$D^{(m,n)}_{l,k}(\varphi' \mp \varphi'')$$
$$= -\mathrm{i} \cos\left[\left(\frac{n-l}{n}\right)(\varphi'' \mp \varphi' + 2\pi k n/m)\right]$$
$$\times \int_0^z [(\mathrm{i})^{l/n} J_{-l/n}(u) - (-\mathrm{i})^{l/n} J_{l/n}(u)] \exp(\mathrm{i} u \cos(\varphi'' \mp \varphi' + 2\pi k n/m)\, \mathrm{d}u. \tag{22}$$

There are $2m$ classical paths in total, exactly half of which are paths with an odd number of reflections at the surfaces of the wedge. From (20) we see that each path contributes, one non-diffractive term and $(n-1)$ diffractive terms. More explicitly, we have

$$K_{n/m}(r'', \varphi'', t; r', \varphi')$$
$$= \sum_{k=1}^{m} \frac{F^{(m,n)}_{l,k}(\varphi'' - \varphi')}{2\pi \hbar \mathrm{i}} \exp\left(\frac{\mathrm{i}}{\hbar} S^{(n)}_{(m,k)}(\varphi'' - \varphi')\right)$$
$$\times \left[\det\left(\frac{\partial^2 S^{(n)}_{(m,k)}(\varphi'' - \varphi')}{\partial^2 r' r''}\right)\right]^{1/2} - \sum_{k=1}^{m} \frac{F^{(m,n)}_{l,k}(\varphi'' + \varphi')}{2\pi \hbar \mathrm{i}}$$
$$\times \exp\left(\frac{\mathrm{i}}{\hbar} S^{(n)}_{(m,k)}(\varphi'' + \varphi')\right)\left[\det\left(\frac{\partial S^{2(n)}_{(m,k)}(\varphi'' + \varphi')}{\partial^2 r' r''}\right)\right]^{1/2} \tag{23}$$

where the modified factor

$$F^{(m,n)}_{l,k}(\varphi'' \mp \varphi') = \frac{1}{n}\left(1 + \sum_{l=1}^{n-1} D^{(m,n)}_{l,k}(\varphi'' \mp \varphi')\right). \tag{24}$$

It is clear that (23) reduces to the 'collapsing cases' [7, 13] only when $n = 1$. However, (23) can still be represented as a sum over classical paths put with modified Van Vleck formula. For an irrational wedge, the propagator (13) cannot be expressed in the above form since there exist an infinite number of classical paths [14].

Studying the diffractive terms mentioned above, we restrict ourselves to the case of $n=2$ for convenience. Using

$$J_{1/2}(u)=\left(\frac{2}{\pi u}\right)^{1/2}\sin(u)\qquad J_{-1/2}(u)=\left(\frac{2}{\pi u}\right)^{1/2}\cos(u)$$

we have, with the help of 3.361-1 and 9.236-1 in [12],

$$\begin{aligned}
F^{(m,2)}_{1,k}&(\varphi''\mp\varphi')\\
&=\tfrac{1}{2}[1+D^{(m,2)}_{1,k}(\varphi''\mp\varphi')]\\
&=\frac{1}{2}\left[1-\mathrm{i}\left(\frac{2\mathrm{i}}{\pi}\right)^{1/2}\cos[\tfrac{1}{2}(\varphi''\mp\varphi'+4\pi k/m)]\right.\\
&\quad\left.\times\int_0^{\infty}\frac{\exp-2u\mathrm{i}\sin^2[(\varphi''\mp\varphi'+4\pi k/m)/2]}{u^{1/2}}\,\mathrm{d}u\right]\\
&=\frac{1}{2}\left[1-\mathrm{i}\left(\frac{2\mathrm{i}}{\pi}\right)^{1/2}\frac{\lambda r'r''}{\sin(\omega t)}\,{}_1F_1\right.\\
&\quad\left.\times\left(\frac{1}{2},\frac{3}{2},\frac{2\lambda r'r''}{\mathrm{i}\sin(\omega t)}\right)\sin^2[(\varphi''\mp\varphi'+4\pi k/m)/2]\right]
\end{aligned}\tag{25}$$

and for $\varphi''=\varphi'$ and $k=m$ (direct classical path)

$$F^{(m,2)}_{1,m}(0)=\tfrac{1}{2}\left[1-2\mathrm{i}\left(\frac{2\mathrm{i}}{\pi}\right)^{1/2}\left(\frac{\lambda r'r''}{\sin(\omega t)}\right)^{1/2}\right].$$

These diffractive terms are very difficult to interpret physically and do not correspond to any classical paths, even with Keller's generalization of that notation to diffractive rays [15].

5. Density matrix and partition function

For imaginary time $t\to-\mathrm{i}\beta\hbar$, we get the density matrix from (13)

$$\begin{aligned}
\rho(r'',r',\beta)&=\left(\frac{2\lambda}{a\pi\sinh(\beta\hbar\omega)}\right)\exp[-\lambda(r'^2+r''^2)\coth(\beta\hbar\omega)]\\
&\quad\times\sum_{l=1}^{\infty}\sin(l\varphi''/a)\sin(l\varphi'/a)I_{l/a}\left(\frac{\lambda r^2}{\sinh(\beta\hbar\omega)}\right).
\end{aligned}\tag{26}$$

Using the table 6.611-1 in [12], the partition function is of the form

$$\begin{aligned}
Z(\beta)&\equiv\int_0^{\infty}r\,\mathrm{d}r\int_0^{a\pi}\rho(r,r;\beta)\,\mathrm{d}\varphi\\
&=\frac{\lambda}{\sinh(\beta\hbar\omega)}\sum_{l=1}^{\infty}\int_0^{\infty}\exp[(-\lambda r^2/2)\coth(\beta\hbar\omega)]I_{l/a}\left(\frac{\lambda r^2}{\sinh(\beta\hbar\omega)}\right)\mathrm{d}r\\
&=\frac{\exp(-\beta\hbar\omega/2a)}{4\sinh(\beta\hbar\omega)\sinh(\beta\hbar\omega/2a)}
\end{aligned}\tag{27}$$

which gives the free energy

$$F(\beta) \equiv -\frac{1}{\beta} \ln Z(\beta) = \frac{\hbar\omega}{2a} + \frac{1}{\beta} \ln [4 \sinh(\beta\hbar\omega) \sinh(\beta\hbar\omega/2a)]. \tag{28}$$

For $\beta \to \infty$, we have

$$F(\beta) \simeq E_{0,1} = \hbar\omega/a + \hbar\omega \tag{29}$$

which is the ground-state energy in the energy spectrum (9).

Writing the Bessel function as a degenerate hypergeometric function (see table 9.238-1 in [12]):

$$J_\nu(z) = \frac{\exp(-iz)}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu {}_1F_1(\nu+\tfrac{1}{2}, 2\nu+1; 2iz) \tag{30}$$

(13) reduces to

$$\begin{aligned} K_a(r'', \varphi'', t; r', \varphi') &= \left(\frac{2\lambda}{a\pi i \sin(\omega t)}\right) \exp\left(\frac{i\lambda}{2\sin(\omega t)} [(r'^2 + r''^2) \right. \\ &\quad \left. \times \cos(\omega t) - 2r'r'']\right) \sum_{l=1}^{\infty} \frac{1}{\Gamma(l/a+1)} \left(\frac{\lambda r' r''}{2i \sin(\omega t)}\right)^{l/a} \sin(l\varphi''/a) \sin(l\varphi'/a) \\ &\quad \times {}_1F_1\left(l/a + \tfrac{1}{2}, 2l/a + 1; \frac{2\lambda r' r''}{i \sin(\omega t)}\right). \end{aligned} \tag{31}$$

Finally we should mention that the exactly solvable propagator (31) belongs to the first group as classified by Inomata [16] and can be evaluated without applying the conventional dimensional extension technique [17], as we expect.

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References

[1] Feynman R P 1948 *Rev. Mod. Phys.* **20** 367
[2] Khandeker D C and Lawande S V 1986 *Phys. Rep.* **137** 115
[3] Gavean B and Schulman L S 1986 *J. Phys. A: Math. Gen.* **19** 1833
[4] Carpio-Bernido M V, Bernido C C and Inomata A 1989 *Path Integrals From meV to MeV* ed V Sa-Yakanit, W Sritrakod, J O Berananda, M C Gutzwiller, A Inomata, S Lundqvist, J R Klander and L S Schulman (Singapore: World Scientific) p 442
[5] Schulman L S 1982 *Phys. Rev. Lett.* **49** 599
[6] Wiegel F W and van Andel P W 1987 *J. Phys. A: Math. Gen.* **20** 627
[7] Crandall R E 1983 *J. Phys. A: Math. Gen.* **16** 513
[8] Cécile Dewitt-Morette, Low S G, Schulman L S and Shiekh A Y 1986 *Found. Phys.* **16** 311
[9] Cheng B K 1989 *Path Integrals From meV to MeV* ed V Sa-Yakanit, W Sritrakod, J O Berananda, M C Gutzwiller, A Inomata, S Lundqvist, J R Klander and L S Schulman (Singapore: World Scientific) p 370
[10] Van Vleck J M 1928 *Proc. Natl. Acad. Sci., USA* **14** 178
[11] Flügge S 1974 *Practical Quantum Mechanics* (Berlin: Springer) p 108

[12] Gradshteyn I S and Ryzhik I M 1980 *Tables of Integrals, Series and Products* (New York: Academic) 8.976-1 and 8.972-1
[13] Akhundova E A, Dodonov V V and Man'ko V I 1985 *J. Phys. A: Math. Gen.* **18** 467
[14] Ziolkowski R W 1986 *J. Math. Phys.* **27** 2271
[15] Keller J B 1958 *Calculus of Variations and Its Applications* ed L M Graves (New York: McGraw-Hill)
[16] Inomata A 1989 *Path Integral From meV to MeV* ed V Sa-Yakanit, W Sritrakod, J O Berananda, M C Gutzwiller, A Inomata, S Lundqvist, J R Klander and L S Schulman (Singapore: World Scientific) p 112
[17] Inomata A 1988 *Path Summation: Achievements and Goals* ed A Ranfagni, V Sa-Yakanit and L S Schulman (Singapore: World Scientific) p 114